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## Brief Paper

Hamiltonian realizations of nonlinear adjoint operators<sup>☆</sup>Kenji Fujimoto<sup>a</sup>, Jacquélien M.A. Scherpen<sup>b, \*</sup>, W. Steven Gray<sup>c</sup><sup>a</sup>Department of Systems Science, Graduate School of Informatics, Kyoto University, Uji, Kyoto 611-0011, Japan<sup>b</sup>Faculty of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, Netherlands<sup>c</sup>Department of Electrical and Computer Engineering, Old Dominion University, Norfolk, VA 23529-0246, USA

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## Abstract

This paper addresses the issue of state-space realizations for nonlinear adjoint operators. In particular, the relationships between nonlinear Hilbert adjoint operators, Hamiltonian extensions and port-controlled Hamiltonian systems are established. Then, characterizations of the adjoints of controllability, observability and Hankel operators are derived from this analysis. The state-space realizations of such adjoint operators provide new insights on singular value analysis and duality issues in nonlinear control systems theory. Finally, a duality between the controllability and observability energy functions is proved. © 2002 Elsevier Science Ltd. All rights reserved.

**Keywords:** Nonlinear systems; State-space realization; Duality; Controllability; Observability

## 1. Introduction

Adjoint operators play an important role in linear control systems theory. They provide a duality between inputs and outputs that is useful in a variety of problems. For example, properties with respect to the input, e.g. controllability and stabilizability, directly convert to the dual concepts for the output, namely observability and detectability. Furthermore, the connection of adjoint operators with singular values is well known (e.g. Zhou, Doyle, & Glover, 1996) and the concept of singular values has been widely used in the last decades for model reduction purposes. For a linear operator (transfer function)  $G(s): E \rightarrow F$  with  $E$  and  $F$  Hilbert spaces, it is well known that if the operator  $G(s)$  has a state-space realization, then its adjoint operator  $G^*(s): F \rightarrow E$  can readily be described by the dual state-space realization. The extension of the latter linear adjoint characterization to nonlinear systems is the main topic of this paper. While our main motivation for studying the nonlinear

adjoint concept is to understand the relation between nonlinear singular value functions and the nonlinear Hankel operator, this concept is of importance for many other nonlinear systems concepts which can be considered as extensions of the linear systems theory.

Nonlinear adjoint operators are available in the mathematics literature (see Scherpen & Gray, 2002). The *nonlinear Hilbert adjoint* operator was introduced in Gray and Scherpen (1998), Scherpen and Gray (1999) as a special class of nonlinear adjoint operators and is defined with the help of the Hilbert inner product. Given a mapping  $\Sigma: E \rightarrow F$  with Hilbert spaces  $E$  and  $F$ , a corresponding Hilbert adjoint of  $\Sigma$  is a mapping  $\Sigma^*: F \times E \rightarrow E$  such that

$$\langle \Sigma(u), y \rangle_F = \langle u, \Sigma^*(y, u) \rangle_E \quad (1)$$

holds for  $\forall u \in E, \forall y \in F$ . Conditions for the existence of such operators were given in Scherpen and Gray (2002), but no state-space realization was given. On the other hand, Hamiltonian extensions (Crouch & Van der Schaft, 1987) are commonly used to characterize state-space adjoints of nonlinear control systems. They are based on the variational system realization. Furthermore, in Scherpen and Van der Schaft (1994), Ball and Van der Schaft (1996) and Van der Schaft (2000), Hamiltonian extensions are used extensively as state-space adjoints of nonlinear systems to characterize norm preserving properties. In fact, in the linear case, the Hamiltonian extension of a given operator with a state-space

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realization is the Hilbert adjoint system. However, in the nonlinear setting, this is not such a straightforward issue, and it is shown in this paper that additional dynamics are required for such a relation.

We first develop a relationship between a nonlinear Hilbert adjoint operator and Hamiltonian extensions, where the introduction of additional dynamics is motivated by comparison with the linear case. Then, in order to give a lower dimensional state-space realization of a nonlinear adjoint operator (which requires less restrictive assumptions), system representations falling in the class of port-controlled Hamiltonian systems (Van der Schaft, 2000; Maschke & Van der Schaft, 1992) are used. Such state-space realizations are shown to be particularly convenient for providing realizations of the observability, controllability and Hankel operators. These operators are important in model reduction theory for nonlinear systems in that they supply a set of singular value functions related to input-state and state-output behaviors (Gray & Scherpen, 1998; Scherpen & Gray, 1999; Scherpen & Gray, 2000; Scherpen & Van der Schaft, 1994; Scherpen, 1993). Finally, a certain type of duality (the duality in the sense of Young (Arnold, 1989)) between the observability and controllability energy functions is introduced. This duality relies on the concept of Legendre transformations.

## 2. State-space realizations of nonlinear adjoint operators

### 2.1. Adjoint operators and Hamiltonian extensions

In order to relate the nonlinear Hilbert adjoint operators defined in (1) to the existing state-space adjoint notion given by Hamiltonian extensions, consider a possibly time-varying continuous input–output system  $\Sigma: L_2^m[t^0, t^1] \rightarrow L_2^r[t^0, t^1]$  defined on a (possibly infinite) time interval  $[t^0, t^1] \subset \mathbb{R}$ . Its state-space realization is given by

$$u \mapsto y = \Sigma(u): \begin{cases} \dot{x} = f(x, u, t), & x(t^0) = 0, \\ y = h(x, u, t) \end{cases} \quad (2)$$

with  $x(t) \in \mathbb{R}^n$ ,  $u(t) \in \mathbb{R}^m$  and  $y(t) \in \mathbb{R}^r$ . It is assumed that  $f$  and  $h$  are smooth and that  $f(0, 0, t) = 0$  and  $h(0, 0, t) = 0$ . Throughout the paper we use an abuse of notation by sometimes leaving out the arguments of functions. The Hamiltonian extension of  $\Sigma$  is given by a Hamiltonian control system (Crouch & Van der Schaft, 1987)

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p}^T = f(x, u, t), & x(t^0) &= 0, \\ \dot{p} &= -\frac{\partial H}{\partial x}^T = -\left(\frac{\partial f}{\partial x}^T p + \frac{\partial h}{\partial x}^T u_a\right), & p(t^1) &= 0, \\ y_a &= \frac{\partial H}{\partial u}^T = \frac{\partial f}{\partial u}^T p + \frac{\partial h}{\partial u}^T u_a \\ y &= \frac{\partial H}{\partial u_a}^T = h(x, u, t) \end{aligned} \quad (3)$$

with the Hamiltonian

$$H(x, p, u, u_a, t) := p^T f(x, u, t) + u_a^T h(x, u, t). \quad (4)$$

We now prove some properties of this system which are related to the nonlinear Hilbert adjoint operator.

**Proposition 1.** Consider the Hamiltonian extension (3) of  $\Sigma$ . Suppose  $f$  and  $h$  are time-invariant, i.e.  $f = f(x, u)$  and  $h = h(x, u)$ . Define scalar-valued functions  $H_1$ ,  $H_2$  and  $H_3$  as

$$H_1 = H - \frac{\partial H}{\partial u_a} u_a, \quad H_2 = H - \frac{\partial H}{\partial u} u,$$

$$H_3 = H - \frac{\partial H}{\partial u} u - \frac{\partial H}{\partial u_a} u_a.$$

Then the following relations hold:

$$\begin{aligned} \frac{dH}{dt} &= y_a^T \dot{u} + y^T \dot{u}_a, & \frac{dH_1}{dt} &= y_a^T \dot{u} - \dot{y}^T u_a \\ \frac{dH_2}{dt} &= -\dot{y}_a^T u + y^T \dot{u}_a, & \frac{dH_3}{dt} &= -\dot{y}_a^T u - \dot{y}^T u_a. \end{aligned} \quad (5)$$

**Proof.** The first equation in (5) follows from

$$\begin{aligned} \frac{dH}{dt} &= \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \dot{p} + \frac{\partial H}{\partial u} \dot{u} + \frac{\partial H}{\partial u_a} \dot{u}_a \\ &= \frac{\partial H}{\partial x} \frac{\partial H}{\partial p}^T - \frac{\partial H}{\partial p} \frac{\partial H}{\partial x}^T + y_a^T \dot{u} + y^T \dot{u}_a = y_a^T \dot{u} + y^T \dot{u}_a. \end{aligned}$$

Then the time derivative of the other functions are obtained straightforwardly.  $\square$

This proposition shows that the Hamiltonian extension has a close relationship to nonlinear Hilbert adjoint operators. For example, the property for  $H_1$  in (5) implies the mapping  $u_a \mapsto y_a$  is the nonlinear Hilbert adjoint of the variational mapping  $\dot{u} \mapsto \dot{y}$ , while the property for  $H_2$  shows that the original mapping  $u \mapsto y$  coincides with the adjoint of the variational map  $\dot{u}_a \mapsto \dot{y}_a$ . This relation can be utilized to derive a state-space realization of a nonlinear Hilbert adjoint operator of an input-affine nonlinear system under some additional assumptions. The property for  $H_3$  in (5) is similar to a basic property of *physical* Hamiltonian control systems, namely the so-called *energy balancing* (Crouch & Van der Schaft, 1987). By studying those properties, it becomes clear that the Hamiltonian extension of  $\Sigma$  is not a state-space realization of the nonlinear Hilbert adjoint of  $\Sigma$ . However, the above proposition does provide a tool to obtain such state-space realization via the Hamiltonian extensions of  $\Sigma$  as can be seen in the next corollary.

**Corollary 2.** Consider the input–output system  $\Sigma$  with state-space realization (2) defined for  $t \in [t^0, t^1] \subset \mathbb{R}$ . Suppose  $f$  and  $h$  are time-invariant and input-affine, i.e.  $f \equiv g_0(x) + g(x)u$  and  $h \equiv k_0(x) + k(x)u$  for some smooth

functions  $g_0$ ,  $g$ ,  $k_0$  and  $k$ . Suppose moreover that

$$u \in L_2^m[t^0, t^1], \quad u_b \in L_2^r[t^0, t^1]$$

$$\Rightarrow \|x(t^1)\| < \infty, \quad \|p_1(t^0)\| < \infty, \quad \|p_2(t^0)\| < \infty \quad (6)$$

holds for the state-space system  $(u_b, u) \mapsto y_b = \Sigma^*(u_b, u)$ :

$$\begin{cases} \dot{x} = g_0(x) + g(x)u, & x(t^0) = 0, \\ \dot{p}_1 = -\frac{\partial g_0}{\partial x}^T p_1 - \frac{\partial k_0}{\partial x}^T p_2, & p_1(t^1) = 0, \\ \dot{p}_2 = u_b, & p_2(t^1) = 0, \\ y_b = \left( \frac{\partial(g^T p_1)}{\partial x} + \frac{\partial(k^T p_2)}{\partial x} \right) g_0(x) \\ \quad - g^T(x) \left( \frac{\partial g_0}{\partial x}^T p_1 + \frac{\partial k_0}{\partial x}^T p_2 \right) + k^T(x)u_b. \end{cases} \quad (7)$$

Then a state-space realization of the nonlinear Hilbert adjoint  $\Sigma^*: L_2^m[t^0, t^1] \times L_2^r[t^0, t^1] \rightarrow L_2^m[t^0, t^1]$  is given by (7).

**Proof.** System (7) is derived by defining  $u_b := \dot{u}_a$ ,  $y_b := \dot{y}_a$  and  $p_2 := u_a$  within the Hamiltonian extension (3). Note that from the input-affine form of  $f$  and  $h$ , we have

$$H_2(x, p_1, p_2) = p_1^T g_0(x) + p_2^T k_0(x).$$

The initial condition of (7) and  $g_0(0) = 0$  and  $k_0(0) = 0$  imply that

$$H_2(x(t^0), p_1(t^0), p_2(t^0)) = H_2(x(t^1), p_1(t^1), p_2(t^1)) = 0.$$

Then it follows from (5) in Proposition 1 that

$$\begin{aligned} 0 &= [H_2]_{t=t^0}^{t=t^1} = \int_{t^0}^{t^1} \frac{dH_2}{dt} dt = \langle y, \dot{u}_a \rangle_{L_2^r[t^0, t^1]} - \langle u, \dot{y}_a \rangle_{L_2^m[t^0, t^1]} \\ &= \langle y, u_b \rangle_{L_2^r[t^0, t^1]} - \langle u, y_b \rangle_{L_2^m[t^0, t^1]}. \end{aligned}$$

Substituting  $y = \Sigma(u)$  and  $y_b = \Sigma^*(u_b, u)$  yields the defining equation of a nonlinear Hilbert adjoint operator

$$\langle \Sigma(u), u_b \rangle_{L_2^r[t^0, t^1]} = \langle u, \Sigma^*(u_b, u) \rangle_{L_2^m[t^0, t^1]}$$

and this completes the proof.  $\square$

Observe that state-space realization (7) has  $(2n + r)$  states, and corresponds in the linear case to  $(s\Sigma(s)(1/s))^* = s\Sigma^*(s)(1/s)$ . Intuitively this follows from the original definition of the Hamiltonian extension as the adjoint of the variational system. Of course this input–output mapping coincides with  $\Sigma^*(s)$  for a linear system, but not generally for nonlinear systems. Furthermore, observe that this realization requires the restrictive assumption (6) because otherwise  $t \in (-\infty, \infty)$  in (6), i.e.  $t^0 \rightarrow -\infty$  and  $t^1 \rightarrow \infty$ , implies the anti-stability of a non-minimum phase operator  $\Sigma^*(s)(1/s)$ . Hence, in general, assumption (6) holds only for a finite time interval  $t \in [t^0, t^1]$ .

## 2.2. Adjoint operators and port-controlled Hamiltonian systems

The state-space realizations of a nonlinear Hilbert adjoint operator based on Hamiltonian extensions given in Corollary 2 are somewhat limited because of the need for restrictive assumptions, i.e., they are available only for input-affine nonlinear systems, they are  $(2n + r)$ -dimensional, and they require a restrictive anti-stability assumption (6). We now produce a more general state-space formulation that is based on the concept of port-controlled Hamiltonian systems (e.g., Van der Schaft, 2000; Maschke & Van der Schaft, 1992).

Consider the input–output system  $\Sigma$  in (2). This system can be regarded as an operator  $\hat{\Sigma}: \mathbb{R}^n \times L_2^m[t^0, t^1] \rightarrow \mathbb{R}^n \times L_2^r[t^0, t^1]$  with a state-space realization

$$(x^0, u) \mapsto (x^1, y) = \hat{\Sigma}(x^0, u): \begin{cases} \dot{x} = f(x, u, t), & x(t^0) = x^0, \\ y = h(x, u, t), \\ x^1 = x(t^1). \end{cases} \quad (8)$$

By a specialized version of the Fundamental Theorem of Integral Calculus (e.g. Milnor, 1963), it is well known that  $f$  and  $h$  can be factorized. These factorizations can be used in combination with the concept of a port-controlled Hamiltonian system to obtain a  $2n$ -dimensional state-space realization of a nonlinear Hilbert adjoint operator of  $\hat{\Sigma}$ , as is done in the following proposition.

**Proposition 3.** Consider the operator  $\hat{\Sigma}: \mathbb{R}^n \times L_2^m[t^0, t^1] \rightarrow \mathbb{R}^n \times L_2^r[t^0, t^1]$  with a state-space realization as defined in (8) and with  $t \in [t^0, t^1] \subset \mathbb{R}$ . Furthermore, consider the corresponding port-controlled Hamiltonian system  $H_{\hat{\Sigma}}: \mathbb{R}^{2n} \times L_2^{m+r}[t^0, t^1] \rightarrow \mathbb{R}^{2n} \times L_2^{m+r}[t^0, t^1]$  given by

$$(x^0, p^1, \hat{u}) \mapsto (x^1, p^0, \hat{y}) = H_{\hat{\Sigma}}(x^0, p^1, \hat{u}): \begin{cases} \dot{\hat{x}} = \hat{J}(\hat{x}, \hat{u}, t) \frac{\partial \hat{H}}{\partial \hat{x}}^T + \hat{g}(\hat{x}, \hat{u}, t) \hat{u}, \\ \hat{y} = \hat{g}^T(\hat{x}, \hat{u}, t) \frac{\partial \hat{H}}{\partial \hat{x}}^T + \hat{D}(\hat{x}, \hat{u}, t) \hat{u}, \\ \hat{x}(t^0) = (x^0, p^0), \\ \hat{x}(t^1) = (x^1, p^1) \end{cases} \quad (9)$$

with  $\hat{x}(t) := (x(t), p(t)) \in \mathbb{R}^n \times \mathbb{R}^n$ ,  $\hat{u}(t) := (u(t), u_a(t)) \in \mathbb{R}^m \times \mathbb{R}^r$ ,  $\hat{y}(t) := (y_a(t), -y(t)) \in \mathbb{R}^m \times \mathbb{R}^r$  and

$$\begin{aligned} \hat{H} &:= p^T x, \quad \hat{J} := \begin{pmatrix} 0 & A \\ -A^T & 0 \end{pmatrix}, \\ \hat{g} &:= \begin{pmatrix} B & 0 \\ 0 & -C^T \end{pmatrix}, \quad \hat{D} := \begin{pmatrix} 0 & D^T \\ -D & 0 \end{pmatrix}. \end{aligned} \quad (10)$$

Here  $A(x, u, t) \in \mathbb{R}^{n \times n}$ ,  $B(x, u, t) \in \mathbb{R}^{n \times m}$ ,  $C(x, u, t) \in \mathbb{R}^{r \times n}$  and  $D(x, u, t) \in \mathbb{R}^{r \times m}$  are appropriate matrices such that

$$f(x, u, t) = A(x, u, t)x + B(x, u, t)u, \quad (11)$$

$$h(x, u, t) = C(x, u, t)x + D(x, u, t)u \quad (12)$$

hold. Suppose that

$$\begin{aligned} \|x^0\| < \infty, \quad \|p^1\| < \infty, \quad u \in L_2^m[t^0, t^1], \quad u_a \in L_2^r[t^0, t^1] \\ \Rightarrow \|x^1\| < \infty, \quad \|p^0\| < \infty. \end{aligned} \quad (13)$$

Then the mapping  $(x^0, p^1, u, u_a) \mapsto (p^0, y_a)$  corresponding to the state-space realization (9) is a state-space realization of a nonlinear Hilbert adjoint operator  $\hat{\Sigma}^* : \mathbb{R}^{2n} \times L_2^m[t^0, t^1] \times L_2^r[t^0, t^1] \rightarrow \mathbb{R}^n \times L_2^r[t^0, t^1]$  of  $\hat{\Sigma}$ .

**Proof.** Since (9) is a time-varying port-controlled Hamiltonian system (see e.g., Fujimoto & Sugie, 1998; Fujimoto, 2000), we have that

$$\begin{aligned} \frac{d\hat{H}}{dt} &= \frac{\partial \hat{H}}{\partial \hat{x}} \left( \hat{J} \frac{\partial \hat{H}}{\partial \hat{x}} + \hat{g}\hat{u} \right) \\ &= \frac{\partial \hat{H}}{\partial \hat{x}} \hat{g}\hat{u} = \left( \frac{\partial \hat{H}}{\partial \hat{x}} \hat{g} + \hat{u}^T \hat{D}^T \right) \hat{u} \\ &= \hat{y}^T \hat{u} = y_a^T u - y^T u_a. \end{aligned}$$

Integrating this equation with  $\hat{H} = p^T x$  yields

$$\begin{aligned} \langle (x^1, y), (p^1, u_a) \rangle_{\mathbb{R}^n \times L_2^r[t^0, t^1]} \\ = \langle (x^0, u), (p^0, y_a) \rangle_{\mathbb{R}^n \times L_2^r[t^0, t^1]}. \end{aligned} \quad (14)$$

Substituting  $(p^0, y_a) = \hat{\Sigma}^*((p^1, u_a), (x^0, u))$  and  $(x^1, y) = \hat{\Sigma}(x^0, u)$  yields

$$\begin{aligned} \langle \hat{\Sigma}(x^0, u), (p^1, u_a) \rangle_{\mathbb{R}^n \times L_2^r} \\ = \langle (x^0, u), \hat{\Sigma}^*((p^1, u_a), (x^0, u)) \rangle_{\mathbb{R}^n \times L_2^m}. \end{aligned}$$

This proves the proposition.  $\square$

It should be noted that the characterization given in the above proposition yields a coordinate dependent state-space characterization of a nonlinear Hilbert adjoint in the sense that if we apply a coordinate transformation, the port-controlled Hamiltonian structure is lost. This is due to the fact that the Hamiltonian given in (10) is intrinsically coordinate dependent. On the other hand, it provides very natural state-space realizations of adjoint operators because it requires rather mild assumptions. The Hamiltonian extension given in the previous subsection is coordinate free, i.e., the Hamiltonian structure is maintained due to the fact that a coordinate transformation is canonical in this case. Of course both the port-controlled Hamiltonian adjoint characterization and the Hamiltonian extension coincide with each other in the linear case. Further, as in Corollary 2,

the following corollary yields a state-space realization for the nonlinear Hilbert adjoint of  $\Sigma$  in (2) when the initial conditions are set to zero.

**Corollary 4.** Consider the system  $\Sigma$  in (2) with the initial condition  $x^0=0$  and let  $\Sigma : L_2^m[t^0, t^1] \rightarrow L_2^r[t^0, t^1]$  denote the mapping  $u \mapsto y$ . Suppose the assumption (13) holds. Then a state-space realization of the nonlinear Hilbert adjoint  $\Sigma^* : L_2^m[t^0, t^1] \times L_2^r[t^0, t^1] \rightarrow L_2^m[t^0, t^1]$  of  $\Sigma$  is given by

$$\begin{aligned} (u_a, u) \mapsto y_a = \Sigma^*(u_a, u): \\ \begin{cases} \dot{x} = f(x, u, t), & x(t^0) = 0, \\ \dot{p} = -A^T(x, u, t)p - C^T(x, u, t)u_a, & p(t^1) = 0, \\ y_a = B^T(x, u, t)p + D^T(x, u, t)u_a. \end{cases} \end{aligned} \quad (15)$$

### 3. Energy functions and operators

#### 3.1. Observability, controllability and Hankel operators

In this section, the results of Section 2 are applied to obtain state-space realizations for nonlinear Hilbert adjoints of some useful operators in nonlinear control theory, i.e., the Hankel, observability and controllability operators. These operators were first introduced in Gray and Scherpen (1998) and Scherpen and Gray (1999), and can be seen as natural extensions from the linear theory (Zhou et al., 1996). Only time-invariant, input-affine systems are considered without direct feedthrough, that is

$$\Sigma: \begin{cases} \dot{x} = f(x) + g(x)u, \\ y = h(x) \end{cases} \quad (16)$$

defined for  $t \in (-\infty, \infty)$ . Assume for the remainder of the paper that  $f(x)$  is asymptotically stable. State-space realizations which describe the observability and controllability operators are given by

$$x^0 \mapsto y = \mathcal{O}(x^0): \begin{cases} \dot{x} = f(x), & x(0) = x^0, \\ y = h(x), \end{cases} \quad (17)$$

$$u \mapsto x^1 = \mathcal{C}(u):$$

$$\begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u), & x(-\infty) = 0, \\ x^1 = x(0). \end{cases} \quad (18)$$

Here  $\mathcal{F}_- : L_2^m[0, \infty) \rightarrow L_2^m(-\infty, 0]$  denotes the so-called *time flipping operator* defined for  $t \leq 0$  by  $\mathcal{F}_-(u)(t) = u(-t)$  and zero otherwise. Furthermore, the Hankel operator  $\mathcal{H} : L_2^m[0, \infty) \rightarrow L_2^r[0, \infty)$  of  $\Sigma$  is given by

$$\mathcal{H} = \mathcal{O} \circ \mathcal{C} = \Sigma \circ \mathcal{F}_- \quad (19)$$

mapping past inputs to future outputs. The original definitions of these operators via Chen–Fliess functional expansions along with more detailed discussions can be found in Gray and Scherpen (1998), Scherpen and Gray (1999) and Gray and Scherpen (2002). State-space realizations for

nonlinear Hilbert adjoint operators of  $\mathcal{O}$ ,  $\mathcal{C}$  and  $\mathcal{H}$  are given as follows.

**Proposition 5.** Consider the operator  $\Sigma$  with state-space realization (16). Suppose that assumption (13) in Proposition 3 holds for the relevant port-controlled Hamiltonian system (9). Then state-space realizations of  $\mathcal{O}^*: L_2^r[0, \infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $\mathcal{C}^*: \mathbb{R}^n \times L_2^m[0, \infty) \rightarrow L_2^m[0, \infty)$  and  $\mathcal{H}^*: L_2^r[0, \infty) \times L_2^m[0, \infty) \rightarrow L_2^m[0, \infty)$  are given by

$$p^0 = \mathcal{O}^*(x^0, u_a): \begin{cases} \dot{x} = f(x), & x(0) = x^0, \\ \dot{p} = -A^T(x)p \\ \quad - C^T(x)u_a, & p(\infty) = 0, \\ p^0 = p(0), \end{cases} \quad (20)$$

$$y_a = \mathcal{C}^*(p^1, u): \begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u), & x(-\infty) = 0, \\ \dot{p} = -A^T(x)p, & p(0) = p^1, \\ y_a = \mathcal{F}_+(g^T(x)p), \end{cases} \quad (21)$$

$$y_a = \mathcal{H}^*(u_a, u): \begin{cases} \dot{x} = f(x) + g(x)\mathcal{F}_-(u), & x(-\infty) = 0, \\ \dot{p} = -A^T(x)p \\ \quad - C^T(x)u_a, & p(\infty) = 0, \\ y_a = \mathcal{F}_+(g^T(x)p) \end{cases} \quad (22)$$

with matrices  $A(x) \in \mathbb{R}^{n \times n}$  and  $C(x) \in \mathbb{R}^{r \times n}$  such that  $f(x) \equiv A(x)x$  and  $h(x) \equiv C(x)x$  hold. Here  $\mathcal{F}_+: L_2^m(-\infty, 0] \rightarrow L_2^m[0, \infty)$  denotes yet another time flipping operator defined for  $t \geq 0$  by  $\mathcal{F}_+(u)(t) = u(-t)$  and zero otherwise.

**Proof.** Substitute  $t^0 = 0$ ,  $t^1 = \infty$ ,  $p^1 = p(\infty) = 0$  and  $u = 0$  into Eq. (14) to obtain

$$\begin{aligned} \langle y, u_a \rangle_{L_2^r[0, \infty)} &= \langle (x^1, y), (0, u_a) \rangle_{\mathbb{R}^n \times L_2^r[0, \infty)} \\ &= \langle (x^0, 0), (p^0, y_a) \rangle_{\mathbb{R}^n \times L_2^m[0, \infty)} = \langle x^0, p^0 \rangle_{\mathbb{R}^n}. \end{aligned}$$

Substituting, moreover,  $y = \mathcal{O}(x^0)$  and  $p^0 = \mathcal{O}^*(x^0, u_a)$  as in (20) yields

$$\langle \mathcal{O}(x^0), u_a \rangle_{L_2^r[0, \infty)} = \langle x^0, \mathcal{O}^*(u_a, x^0) \rangle_{\mathbb{R}^n}.$$

This proves the first part. The second part can be proven in a similar way. Substituting  $t^0 = -\infty$ ,  $t^1 = 0$ ,  $x^0 = x(-\infty) = 0$  and  $u_a = 0$  into Eq. (14) yields

$$\begin{aligned} \langle x^1, p^1 \rangle_{\mathbb{R}^n} &= \langle (x^1, y), (p^1, 0) \rangle_{\mathbb{R}^n \times L_2^r(-\infty, 0]} \\ &= \langle (0, \mathcal{F}_-(u)), (p^0, y_a) \rangle_{\mathbb{R}^n \times L_2^m(-\infty, 0]} \\ &= \langle \mathcal{F}_-(u), y_a \rangle_{L_2^r(-\infty, 0]} = \langle u, \mathcal{F}_+(y_a) \rangle_{L_2^m[0, \infty)}. \end{aligned} \quad (23)$$

Then letting  $x^1 = \mathcal{C}(u)$  and  $y_a = \mathcal{C}^*(p^1, u)$  in (21) yields

$$\langle \mathcal{C}(u), p^1 \rangle_{\mathbb{R}^n} = \langle u, \mathcal{C}^*(p^1, u) \rangle_{L_2^m(-\infty, 0]}.$$

This proves the second part. The last part is proven by noting that Eq. (23) implies that  $\mathcal{F}_-^* = \mathcal{F}_+$  and  $\mathcal{F}_+^* = \mathcal{F}_-$ . Combining this with the linear adjoint property of the time flipping operators, the results of Scherpen and Gray, 2002, and the definition of the Hankel operator given by (19) gives

$$\begin{aligned} \mathcal{H}^*(u_a, u) &= (\Sigma \circ \mathcal{F}_-)^*(u_a, u) = \mathcal{F}_-^* \circ \Sigma^*(u_a, \mathcal{F}_-(u)) \\ &= \mathcal{F}_+ \circ \Sigma^*(u_a, \mathcal{F}_-(u)). \end{aligned}$$

This completes the proof.  $\square$

### 3.2. Observability and controllability functions

In this subsection, duality relationships among the observability and controllability functions, operators and nonlinear Gramian extensions are discussed. We begin with the definition of the energy functions, also referred to as the observability and controllability functions.

**Definition 6** (Scherpen, 1993). The observability function,  $L_o(x)$ , and the controllability function,  $L_c(x)$ , of  $\Sigma$  as in (16) are defined by

$$L_o(x^0) := \frac{1}{2} \int_0^\infty \|y(t)\|^2 dt, \quad x(0) = x^0, \quad u(t) \equiv 0, \quad (24)$$

$$L_c(x^1) := \min_{\substack{u \in L_2^m(-\infty, 0] \\ x(-\infty)=0, x(0)=x^1}} \frac{1}{2} \int_{-\infty}^0 \|u(t)\|^2 dt. \quad (25)$$

It is assumed throughout that  $L_o$  and  $L_c$  exist and are smooth. These functions are closely related to observability and controllability operators and Gramians (Gray & Scherpen, 1998; Scherpen & Gray, 1999; Gray & Scherpen, 2002). Specifically, the relations between the observability function, operator and “Gramian” are given by

$$\begin{aligned} L_o(x^0) &= \frac{1}{2} \|\mathcal{O}(x^0)\|_{L_2^r}^2 = \frac{1}{2} \langle x^0, \mathcal{O}^*(\mathcal{O}(x^0), x^0) \rangle_{\mathbb{R}^n} \\ &= \frac{1}{2} \langle x^0, p^0 \rangle_{\mathbb{R}^n} =: \frac{1}{2} \langle x^0, \phi(x^0) \rangle_{\mathbb{R}^n}. \end{aligned} \quad (26)$$

Here,  $p^0 = p(0)$  is the initial state of the state-space realization of  $\mathcal{O}^*$  in (20) with input  $(x^0, u_a) = (x^0, \mathcal{O}(x^0))$ . The function  $\phi(x^0)$  can always be expressed as  $\phi(x^0) = Q(x^0)x^0$  using a square symmetric matrix  $Q(x^0)$ . In the linear case this matrix equals the observability Gramian. Furthermore, note that the relation

$$L_o(x(t)) = \frac{1}{2} \langle x(t), p(t) \rangle_{\mathbb{R}^n} \quad (27)$$

holds along the trajectory of the state-space realization (20) of  $\mathcal{O}^*(x^0, \mathcal{O}(x^0))$ . Particularly, in the case where  $x(t) \in \mathbb{R}$ , the function  $\phi$  in Eq. (26) is readily computed as follows.

**Example 7.** Consider the system (16) with  $n = m = r = 1$ , i.e.,  $x(t), u(t), y(t) \in \mathbb{R}$ . The observability operator and its Hilbert adjoint are given by (17) and (20), respectively, with the unique decomposition  $A(x) = f(x)/x$  and  $C(x) = h(x)/x$ . Since  $\phi$  is unique in the case  $n = 1$ , it follows from (26) and (27) that the states of  $\mathcal{O}^*(x^0, \mathcal{O}(x^0))$  satisfy  $(x(t), p(t)) = (x(t), \phi(x(t)))$  for  $\forall t \in [0, \infty)$  with a scalar-valued function  $\phi: \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $\phi(0) = 0$ . Therefore

$$\dot{p} = -\frac{f(x)}{x} \phi(x) - \frac{h(x)}{x} h(x) \equiv \frac{d\phi(x)}{dx} \dot{x} = \frac{d\phi(x)}{dx} f(x).$$

The solution of this equation is given by

$$\phi(x) = -\frac{1}{x} \int_0^x \frac{h(\xi)^2}{f(\xi)} d\xi.$$

As explained above, the function  $(\phi(x)/x)$  is the nonlinear extension of the observability Gramian. Furthermore, the function  $\phi$  can be used to calculate  $L_o$  via (26). For example, take  $f(x) = -x$  and  $h(x) = x^2$ , then it is straightforwardly computed that  $\phi(x) = (1/4)x^3$ , and  $L_o(x) = (1/8)x^4$ .

In the controllability case, there does not exist as nice a relation for  $\mathcal{C}^*$ . Nevertheless, we are able to obtain a duality result along the linear lines of thinking in the following proposition. It concerns a duality between the observability and controllability functions.

**Proposition 8.** Consider the system  $\Sigma$  with the state-space realization (16). It is assumed that  $f(x)$  is asymptotically stable and that  $L_o(x)$  and  $L_c(x)$  exist and are smooth. Consider the system

$$\begin{aligned} \dot{p} &= A^T(\phi_i(p))p + C^T(\phi_i(p))u_a, \\ y_a &= g^T(\phi_i(p))p \end{aligned} \quad (28)$$

with the subscript  $i \in \{c, o\}$ . Let  $x = \phi_c(p)$  denote the inverse mapping of  $p = (\partial L_c(x)/\partial x)^T$ . Suppose that (28) has observability function  $\tilde{L}_o(p)$  and that  $i = c$ . Then  $\tilde{L}_o(p)$  is given by the Legendre transformation

$$\tilde{L}_o(p) = -L_c(x) + p^T x. \quad (29)$$

Let  $x = \phi_o(p)$  denote the inverse mapping of  $p = (\partial L_o(x)/\partial x)^T$ . Suppose that (28) has controllability function  $\tilde{L}_c(p)$  and that  $i = o$ . Then  $\tilde{L}_c(p)$  is given by the Legendre transformation

$$\tilde{L}_c(p) = -L_o(x) + p^T x. \quad (30)$$

**Proof.** It follows from Scherpen (1993) that the controllability function  $L_o(x)$  of the system  $\Sigma$  is the unique anti-stabilizing solution of the Hamilton–Jacobi equation

$$\frac{\partial L_c}{\partial x}(x)f(x) + \frac{1}{2} \frac{\partial L_c}{\partial x}(x)g(x)g^T(x) \frac{\partial L_c}{\partial x}(x) = 0, \quad (31)$$

and the observability function  $\tilde{L}_o(p)$  of (28) for  $i = c$  is the unique solution of the Lyapunov equation

$$\begin{aligned} \frac{\partial \tilde{L}_o}{\partial p}(p)A(\phi_c(p))^T p \\ + \frac{1}{2} p^T g(\phi_c(p))g^T(\phi_c(p))p = 0. \end{aligned} \quad (32)$$

Here  $x = \phi_c(p)$  is the inverse mapping of  $p = (\partial L_c(x)/\partial x)^T$ . By applying this transformation it follows that (31) and (32) are the same. And thus, a solution for (32) is given by the Legendre transformation (29) which has the property that  $(\partial \tilde{L}_o(p)/\partial p) = \phi_c^T(p)$  holds. Furthermore, this is the observability function of the  $p$ -subsystem because the Lyapunov equation has a unique solution. This proves the first part. The second part can be proven in a similar way.  $\square$

Thus, we can prove a certain type of duality between input and output. This is similar to the linear case. Legendre transformations define for physical Hamiltonian systems the physical dual coordinates. However, due to the difference between the adjoint notion and the duality result above, it follows that duality does not hold in the exact same way for the general case, although some similarity with physical systems is retained. Legendre transformations transform a convex function into another convex function, which is necessary for relating the controllability and observability of “dual” dynamics with the original dynamics. The type of duality of Proposition 8 is the so-called *duality in the sense of Young* (Arnold, 1989), which often appears in the literature of both optimal control (Young, 1969) and optimization theory (Walsh, 1975).

## 4. Conclusion

This paper presented state-space realizations for nonlinear adjoint operators. In particular, the relationships between nonlinear Hilbert adjoint operators, Hamiltonian extensions and port-controlled Hamiltonian systems were established. A characterization of the controllability, observability, and Hankel adjoint operators and functions was then derived using these results. Finally, the duality in the sense of Young was shown to apply to controllability and observability functions in the nonlinear setting.

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